XVI. NETWORK SYNTHESIS

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A. THE NUMBER OF NATURAL FREQUENCIES IN RLC NETWORKS

In networks containing only two kinds of elements (RL, RC, or LC) the number of natural frequencies can be readily determined from the topology of the subgraphs for each kind of element (1). We shall show how the number of natural frequencies of any RLC network can also be determined in a circumspect way. The two-element case is simply a special case of this general method. [The independent work of R. B. Adler (2) and J. Otterman (3) on the subject of this report has been brought to the author's attention.]

Consider an RLC network in a force-free state. At any "solder" or "plier" entry into N (Fig. XVI-1) the voltage or current, respectively, will satisfy a linear differential equation of order p with constant coefficients

$$\left(\frac{d^p}{dt^p} + a_{p-1}\frac{d^{p-1}}{dt^{p-1}} + \dots + a_o\right) \begin{matrix} e(t) \\ \text{or} \\ i(t) \end{matrix} = 0 \tag{1}$$

The solution of this differential equation is of the form

$$\begin{matrix} e(t) \\ \text{or} \\ i(t) \end{matrix} = \sum_k A_k e^{s_k t} \qquad k = 1, 2, \dots, p \tag{2}$$

where the natural frequencies s_k are the roots of the characteristic equation

$$s^p + a_{p-1} s^{p-1} + \dots + a_o = 0 \tag{3}$$

Hence the number of natural frequencies of the system is equal to the degree p of the characteristic equation. The independent constants A_k are determined by the initial state of the system. Since, in general, the number of independent A_k constants is equal to p, the number of natural frequencies is equal to the number of independent initial conditions that can be specified for the system.

The initial state of an RLC system is completely specified when the currents through the inductances and voltages on capacitances are specified at the initial instant. The maximum number of initial conditions that could be specified would equal the number of energy-storing elements. However, the topology of the system will usually impose constraints among some of the voltages on the capacitances and also among some of the currents through the inductances. Each independent linear constraint relation diminishes the number of independent initial conditions by one. Hence the number of independent initial conditions that can be specified (which is equal to the degrees of freedom of the system) is equal to the number of energy-storing elements diminished by the number of

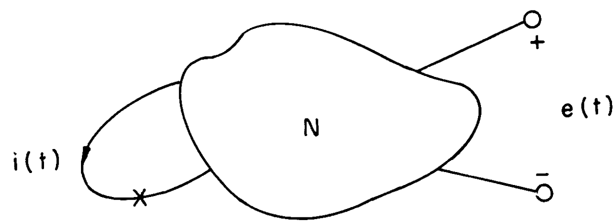

Fig. XVI-1. RLC network with "plier" and "solder" entries for observing the natural behavior of the network.

independent linear constraint relations of voltage and current that exist for the energy-storing elements.

We now proceed to formalize our argument.

Let

b = number of branches in N

b_L = number of L branches in N

b_C = number of C branches in N

Then

$$b_{LC} = b_L + b_C \tag{4}$$

is the total number of energy-storing elements.

Linear constraint relations (1) can exist among energy-storing elements of one kind only. These constraints arise from the topology of the system and can be of two general forms: tie-sets and cut-sets (3).

Let

ℓ_L = number of independent L tie-sets

ℓ_C = number of independent C tie-sets

Then

$$\ell_s = \ell_L + \ell_C \tag{5}$$

is the number of superfluous energy-storing branches arising from independent L and C tie-sets. Similarly, let

n_L = number of independent L cut-sets

n_C = number of independent C cut-sets

Then

$$n_s = n_L + n_C \tag{6}$$

is the number of superfluous energy-storing branches arising from independent L and C cut-sets. The total number of independent linear constraint relations is

$$b_s = \ell_s + n_s \tag{7}$$

and hence the number of independent initial conditions that can be specified (which is equal to the number of natural frequencies) is

$$p = b_{LC} - b_s \tag{8}$$

In order to find the number of independent L and C tie-sets and cut-sets, proper attention must first be given to the effects of the resistance elements in the system.

Figure XVI-2 shows a part of the system N where some inductances and resistances are present. In Fig. XVI-2a the inductances form a tie-set. The resistances that shunt the inductances have no effect on the linear constraint between the currents of the inductance tie-set. This is easily understood by noting that the shunt resistances can be absorbed in networks N_1 and N_2, as shown by the dotted lines, without destroying the tie-set character of the inductances. On the other hand, Fig. XVI-2b shows that because of the series resistance, the inductances here do not form a tie-set, and therefore no linear constraint exists among the currents of these inductances. The same arguments apply to the effect of resistance in capacitance tie-sets. It follows that all tie-sets of L and C can be found from the network with all resistances replaced by open circuits.

Figure XVI-3 shows part of the system N where some capacitances and resistances appear. From Fig. XVI-3a we note that the resistances in series with the capacitances can be absorbed in networks N_1 and N_2 without destroying the cut-set character of the capacitances. In Fig. XVI-3b, however, because of the shunt resistances, the capacitances do not form a cut-set. Again, the same arguments apply to the effect of resistance in cut-sets of inductances. Hence, all cut-sets of L and C can be found from the network with all resistances replaced by short circuits.

The number of independent L and C tie-sets and cut-sets can now be found from appropriate subgraphs as follows:
For any network graph, let

b = number of branches

n_t = number of nodes

s = number of separate parts

Then (ref. 4)

$n_t - s = n =$ number of independent cut-sets

$b - n = \ell =$ number of independent tie-sets

I. With all resistances in the network open-circuited, form an inductance subgraph (L^{∞}-subgraph) by removing all capacitances, and a capacitance subgraph (C^{∞}-subgraph) by removing all inductances. We have

L^{∞}-subgraph	C^{∞}-subgraph
$b_L^{\infty};\ n_{tL}^{\infty};\ s_L^{\infty}$	$b_C^{\infty};\ n_{tC}^{\infty};\ s_C^{\infty}$
$\ell_L^{\infty} = b_L^{\infty} - \left(n_{tL}^{\infty} - s_L^{\infty}\right)$	$\ell_C^{\infty} = b_C^{\infty} - \left(n_{tC}^{\infty} - s_C^{\infty}\right)$

The number of independent L tie-sets is

$$\ell_L = \ell_L^{\infty}$$

The number of independent C tie-sets is

$$\ell_C = \ell_C^{\infty}$$

Hence, by Eq. 5,

$$\ell_s = \ell_L^{\infty} + \ell_C^{\infty} \tag{9}$$

II. With all resistances in the network short-circuited, form a graph of the network (N^{o}-graph), and again subgraphs of inductance (L^{o}-subgraph) and capacitance (C^{o}-subgraph). We have

N^{o}-graph

$$b^{o};\ n_t^{o};\ s^{o}$$

$$n^{o} = n_t^{o} - s^{o}$$

L^{o}-subgraph	C^{o}-subgraph
$b_L^{o};\ n_{tL}^{o};\ s_L^{o}$	$b_C^{o};\ n_{tC}^{o};\ s_C^{o}$
$n_L^{o} = n_{tL}^{o} - s_L^{o}$	$n_C^{o} = n_{tC}^{o} - s_C^{o}$

The number of independent L cut-sets is

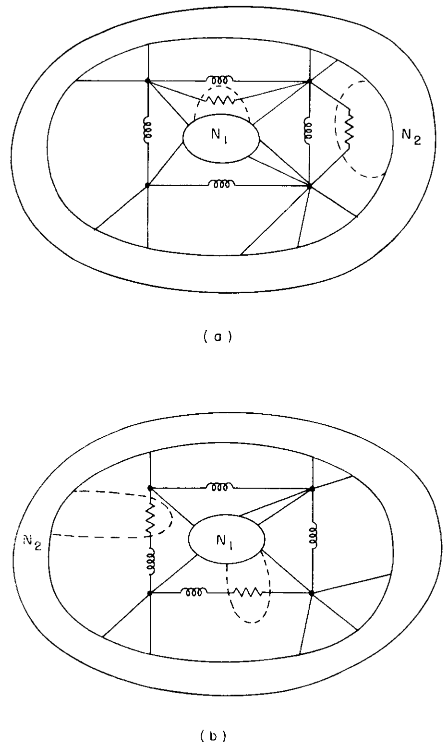

Fig. XVI-2. Effect of resistance on tie-sets: (a) Inductances form a tie-set; (b) inductances do not form a tie-set.

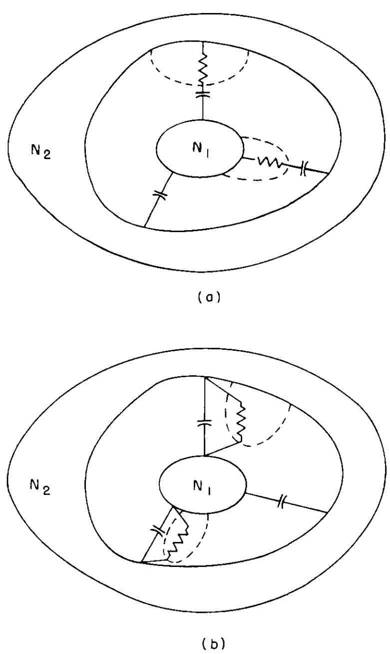

Fig. XVI-3. Effect of resistance on cut-sets: (a) Capacitances form a cut-set; (b) capacitances do not form a cut-set.

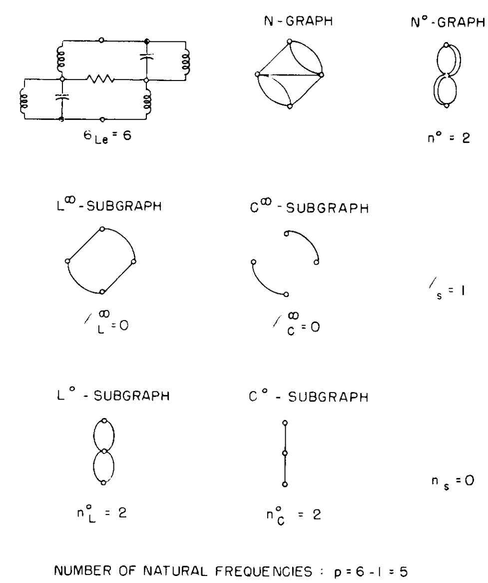

Fig. XVI-4. Example of finding the natural frequencies of an RLC network.

$$n_L = n^o - n_C^o$$

The number of independent C cut-sets is, similarly,

$$n_C = n^o - n_L^o$$

Hence, by Eq. 6,

$$n_s = \left(n^o - n_C^o\right) + \left(n^o - n_L^o\right) \tag{10}$$

Equations 9 and 10 together with Eqs. 7 and 8 give the number of natural frequencies of the complete network.

For networks that contain only two kinds of elements, we can readily show that the present results reduce to those of Guillemin (1). Thus, for LC networks,

$$b_{LC} = b = n + \ell$$

$$p_{LC} = 2(n - n_s) = 2(\ell - \ell_s) \tag{11}$$

where b, n, and ℓ refer to the network graph, and n_s and ℓ_s can be found from the L and C subgraphs.

A simple, nontrivial example of the use of the technique presented for an RLC network is shown in Fig. XVI-4. It is to be noted that if we either short-circuit or open-circuit the resistance, and then determine the number of natural frequencies for the resulting LC network (e.g., with the aid of Eq. 11) we find p = 4. For this simple network, impedance inspection techniques readily show that there are five natural frequencies for the complete network.

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References

1. E. A. Guillemin, Synthesis of Passive Networks (John Wiley and Sons, Inc., New York, 1957), Chap. 6.
2. R. B. Adler, private communication.
3. J. Otterman, On the order of the differential equation describing an electrical network, Proc. IRE 45, 1024 (July 1957).
4. E. A. Guillemin, Introductory Circuit Theory (John Wiley and Sons, Inc., New York, 1953), Chap. 1.